FLEXIBLE MULTIBODY SYSTEMS WITH LARGE DEFORMATIONS USING ABSOLUTE NODAL COORDINATES FOR ISOPARAMETRIC SOLID BRICK ELEMENTS

Lars Kübler*
Institute of Applied Mechanics
University of Erlangen-Nuremberg
Egerlandstr. 5
91058 Erlangen, Germany
kuebler@ltm.uni-erlangen.de

Peter Eberhard
Institute B of Mechanics
University of Stuttgart
Paffenwaldring 9
70550 Stuttgart, Germany
eberhard@mechb.uni-stuttgart.de

Johannes Geisler
Institute of Applied Mechanics
University of Erlangen-Nuremberg
Egerlandstr. 5
91058 Erlangen, Germany
geisler@ltm.uni-erlangen.de

ABSTRACT
In this paper a formulation for flexible Multibody Systems (MBS) is proposed where flexible bodies are described using absolute coordinates for isoparametric brick elements. The use of absolute coordinates allows for large deformations and provides an accurate description of rigid body motion and inertia in the case of large rotations without additional considerations. Further, constant mass matrices are obtained for isoparametric elements.

Brick elements are important, e.g. if general solid bodies with low stiffness, i.e. not negligible large deformations, are part of the MBS and cannot be modeled using beam, plate, or shell elements. Since only nodal translational degrees of freedom are used for brick elements additional questions arise. For example, imposing joint constraints for relative rotations between two bodies requires a nodal reference frame at connection points. An approach is proposed to define such a reference system utilizing displacement information of three finite element nodes.

1 INTRODUCTION
An important aspect within the modeling of flexible MBS is the selection of the finite element formulation for the description of the flexible bodies, where several methodologies have been proposed, [1]. A widely used method is the floating frame of reference formulation, where the configuration of a flexible body is described by the position and orientation of a reference coordinate system and the deformation relative to this reference frame. However, this method is limited to problems with small deformations. For the dynamic analysis and simulation of flexible MBS with large deformations a finite element approach, called absolute nodal coordinate formulation, has been proposed recently, see [1], [2], [3] or [4]. In this method global nodal displacement coordinates and slopes are used in order to define the element configuration. This leads to an important advantage for elements with nodal rotational degrees of freedom, such as beams, plates and shells, where also in the case of large rotations exact modeling of the rigid body dynamics is obtained.

In this paper a formulation for flexible MBS is proposed where flexible bodies are described using absolute coordinates for isoparametric brick elements, [5]. Despite the fact that these elements do not have rotational degrees of freedom, the use of absolute coordinates still provides many advantages. A floating frame of reference formulation leads to highly nonlinear mass matrices resulting from inertia coupling between rigid body motion and elastic deformation and it is limited to small deformations. Other methods like incremental finite element formulations or large rotation vector formulations suffer from different problems, see e.g. [2]. Using absolute coordinates, however, allows for large deformations and preserves an accurate description of rigid body motion and inertia, also in the case of large rotations. Further, with absolute coordinates constant mass matrices are obtained for isoparametric elements [1].

* Address all correspondence to this author.
If general solid bodies with low stiffness, i.e. not negligible large deformations, cannot be modeled using beam, plate, or shell elements, hexagonal elements are important. For this purpose also a compressible neo-Hooke material law is implemented, see [6, 7]. Since only nodal translational degrees of freedom are used for brick elements additional questions arise compared to the other element types mentioned above. For example, imposing joint constraints for relative rotations between two bodies requires additional considerations. A possibility is discussed in this paper to define a nodal reference frame at connection points, where the coordinate system is defined using the position of three nodes of an element.

In Section 2 the modular organization with subsystems of our flexible MBS approach is illustrated. Local descriptions of the different subsystems are determined in Section 3. A method to describe relative orientations of arbitrary marker frames on flexible bodies for hexagonal elements is derived in Section 4. In Section 5, the global equations of motion are determined by assembling the subsystems under application of constraint equations. Here the formulation of the constraint equations for joints attached to flexible bodies is emphasized. Finally, in Section 6 the proposed method is verified by numerical simulation of a 3D mechanism with our flexible MBS program system.

2 MODULAR STRUCTURE OF THE FLEXIBLE MBS

In our approach the global flexible MBS is organized in modules, as illustrated in Figure 1. Several subsystems that can be rigid bodies, flexible bodies or loopfree rigid multibody system substructures are assembled to the global model. Their motion is kinematically constrained by mechanical joints or kinematic drivers, both described by nonlinear algebraic constraint equations. The motion of the elements within the rigid MBS substructures is described by using relative coordinates, while the relative motion of the subsystems is described in Cartesian space.

This approach utilizes on the one hand the efficiency of a formulation with relative coordinates, i.e. description of substructures with minimal coordinates, see e.g. [8]. On the other hand the approach allows for a high level of generality by assembling the subsystems under application of a constraint formulation with the advantage that the formulation of the equations of motion even for complex systems is straightforward and it is open to the addition of various complex system components [1]. This feature proofs for example to be very advantageous when flexible bodies are interconnected with other subsystems, where it is only necessary to provide the finite element description of a free body with absolute nodal coordinates, Section 3.2. Complex relations like inertia coupling between rigid substructures and flexible bodies are considered implicitly via constraint equations.

3 DESCRIPTION OF THE SUBSYSTEMS

In this section local equations of motion of rigid MBS substructures and single flexible bodies are provided. Focus is given to the finite element formulation with absolute nodal coordinates for isoparametric hexagonal elements.

3.1 Rigid MBS Description with Relative Coordinates

The multibody system approach with relative coordinates is described in detail in [8]. Relative coordinates lead for tree-like configurations to a minimum number of ordinary differential equations, using a set of independent generalized coordinates \( y \in \mathbb{R}^{f} \) corresponding to the degree of freedom \( f \) of the MBS. This is a main advantage over the approach with Cartesian coordinates where differential algebraic equations (DAE) of often much higher dimension have to be solved.

A spatial MBS consisting of \( n_b \) bodies with \( n_c \) holonomic constraints, and hence \( f = 6n_b - n_c \) degrees of freedom, can be described, applying for example d’Alembert’s principle, Hamilton’s principle or the Newton-Euler formalism [8] to the balances of linear and angular momentum. This yields the equations of motion

\[
M(t, y) \ddot{y} + k(t, y, \dot{y}) = g(t, y, \dot{y})
\]  

with the symmetric, positive definite mass matrix \( M \in \mathbb{R}^{f \times f} \), the generalized centrifugal and Coriolis forces \( k \in \mathbb{R}^{f} \) and the generalized applied forces \( g \in \mathbb{R}^{f} \).

3.2 Nonlinear Finite Element Description of Flexible Bodies

Flexible bodies are described utilizing a nonlinear finite element (FE) approach with absolute nodal coordinates, i.e. all
nodal coordinates are given in the global inertial system. Non-linearities considered in this formulation are large deformations, large rotations and a hyper-elastic material description. In this section the FE formulation is discussed rather briefly. For further particulars the authors refer the reader for example to [6] or [7].

**Required Quantities from Continuum Mechanics**

A Lagrangian description is chosen for the FE formulation, where all quantities are expressed with respect to the reference configuration with domain \( \Omega_0 \), as illustrated in Figure 2.

![Figure 2. REFERENCE AND CURRENT CONFIGURATION](image)

A point \( P(t_0) \) in the reference configuration maps onto the point \( P(t) \) in the current configuration. Utilizing the displacement field \( u(X, t) \), which contains rigid body translation and rotation and the body deformation, the transition from reference to current configuration is uniquely described by

\[
x(X, t) = X + u(X, t).
\]

Consistent to the Lagrangian description, strains are described by the Green–Lagrangian strain tensor \( \mathbf{E} \)

\[
\mathbf{E} = \frac{1}{2} (\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}),
\]

with the deformation gradient \( \mathbf{F} \) defined from \( dx = \mathbf{F} \cdot dx \).

A conjugated stress measure with respect to the Green–Lagrangian strain tensor is given by the 2\textsuperscript{nd} Piola–Kirchhoff stress tensor \( \mathbf{S} \), which is an objective and symmetric tensor. The dependence of \( \mathbf{S} \) on other stress measures follows from

\[
\mathbf{S} = \mathbf{F}^{-1} \cdot \mathbf{P} \quad \text{with} \quad \mathbf{P} = \text{det}(\mathbf{F}) \mathbf{T} \cdot \mathbf{F}^{-T}
\]

where \( \mathbf{P} \) is the 1\textsuperscript{st} Piola–Kirchhoff stress tensor, which is neither symmetric nor objective, and \( \mathbf{T} \) is the Cauchy stress tensor. The relations in (4) are, e.g., useful to determine the Cauchy stresses, corresponding to the current configuration, for output reasons, whereas \( \mathbf{S} \) is derived from the constitutive equations, given in following paragraph.

**Constitutive Relations**

A hyper-elastic, i.e. nonlinear elastic material description is utilized. For hyper-elastic materials, the constitutive equations for the 2\textsuperscript{nd} Piola–Kirchhoff stress tensor \( \mathbf{S} \) can be written in terms of a strain energy potential \( W(\mathbf{E}) \), see [6] or [7]

\[
\mathbf{S} = \frac{\partial W(\mathbf{E})}{\partial \mathbf{E}} = 2 \frac{\partial W(\mathbf{C})}{\partial \mathbf{C}},
\]

which depends on the Green–Lagrangian strain tensor \( \mathbf{E} \), or rather the right Cauchy–Green tensor \( \mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} = \mathbf{I} + 2 \mathbf{E} \). For isotropic materials the potential can be completely described by the basic invariants of \( \mathbf{C} \)

\[
I_C = \text{tr}(\mathbf{C}), \quad II_C = \frac{1}{2} (\text{tr}(\mathbf{C})^2 - \text{tr}(\mathbf{C} \cdot \mathbf{C})), \quad III_C = \text{det}(\mathbf{C}).
\]

As a specific type of hyper–elastic materials here compressible neo-Hooke material is used, with the strain energy potential [7]

\[
W(I_C; III_C) = \frac{\mu}{2} (I_C - 3) - \mu \ln(\sqrt{III_C}) + \frac{\lambda}{2} (\sqrt{III_C} - 1)^2,
\]

where \( \mu \) and \( \lambda \) are the Lamé constants. Then, it follows from (5) after some transformations

\[
\mathbf{S} = \mu \mathbf{I} + (\lambda (III_C - \sqrt{III_C}) - \mu) \mathbf{C}^{-1}.
\]

**Finite Element Discretization**

For isoparametric solid elements only translations appear for nodal displacements. Arbitrary displacements \( u(X, t) \) are approximated by a set of \( n \) shape functions from the nodal displacements \( U_i \), summarized in vector \( \mathbf{U} \)

\[
\mathbf{u} = \mathbf{N} \cdot \mathbf{U}, \quad \delta \mathbf{u} = \mathbf{N} \cdot \delta \mathbf{U}
\]

with

\[
\mathbf{N} = \begin{bmatrix} N_1 & 0 & 0 & 0 \cdots & 0 & 0 \ 0 & N_1 & 0 & 0 \cdots & 0 & 0 \ 0 & 0 & N_1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 3n},
\]

\[
\mathbf{U} = [U_1 \ U_2 \ldots \ U_n] \in \mathbb{R}^{3n}.
\]

Copyright © 2003 by ASME
The interpolation functions $N_i$ for linear hexagonal elements are given by the following displacement shape functions that relate generic displacements $\hat{r} = [\hat{\xi}, \hat{\eta}, \hat{\zeta}]$ in the reference domain $\Omega_r$ to nodal displacements

$$N_i(\hat{\xi}, \hat{\eta}, \hat{\zeta}) = \frac{1}{8}(1 + \hat{\xi}_i \hat{\eta})(1 + \hat{\eta}_i \hat{\zeta})(1 + \hat{\zeta}_i \hat{\xi}), \quad i = 1, 2, \ldots, 8, \quad (12)$$

where $\hat{\xi}_i$ are the positions of the nodes in the generic configuration.

Transformations between the generic element domain $\Omega_r$ and the real element domain $\Omega_e$, which is part of the reference configuration $\Omega_0$ in a Lagrangian description, are described by the Jacobian matrix $J$

$$J = \frac{\partial \mathbf{X}}{\partial \hat{\mathbf{X}}} = \frac{\partial (\mathbf{N} \cdot \hat{\mathbf{X}})}{\partial \hat{\mathbf{X}}}. \quad (13)$$

with the nodal position vector in the reference configuration $\hat{\mathbf{X}}$ and the relation $\mathbf{X} = \mathbf{N} \cdot \hat{\mathbf{X}}$ that follows for isoparametric elements in correspondence to (9).

**Local Equations of Motion** Basis of the nonlinear FE description is the principle of virtual work, yielding after FE discretization the nonlinear weak form in total Lagrangian formulation, see [7], here given in index notation

$$\delta U_j \left( M_{jk} \Delta U_k - g_j + \int_{\Omega_0} B_{ijk} F_{jk} \Delta s_{ik} dV \right) = 0, \quad (14)$$

with the mass matrix $\mathbf{M}$, the vector of applied forces $\mathbf{g}$ and the operator $B_{ijk}$

$$M_{jk} := \int_{\Omega_0} \rho_0 N_i \dot{N}_i dV, \quad (15)$$

$$g_j := \int_{\Omega_0} \dot{N}_i \rho_0 h_i dV + \int_{\Gamma_0} \dot{N}_i p_i dA, \quad (16)$$

$$B_{ijk} := \frac{\partial N_i}{\partial X_k}. \quad (17)$$

In equation (14) the deformation gradient $\mathbf{F}$ can be determined from

$$F_{ij} = \delta_{ij} + B_{ijk} U_k, \quad (18)$$

and $\mathbf{S}$ follows from (8). The integrals over domain $\Omega_0$ are evaluated from the corresponding integrations over the element domains $\Omega_e$, respectively the generic element domains $\Omega_r$, utilizing the direct stiffness method.

Note that due to integration with respect to the reference configuration the mass matrix $\mathbf{M}$, the B–operator and also the Jacobian matrix $J$ are constant and have to be computed only once. This favorable property appears also for the global flexible MBS approach, since flexible bodies are considered using absolute nodal coordinates with respect to the global inertial system. However, $\mathbf{S}$ in (14) is highly non-linear.

**4 DESCRIPTION OF RELATIVE ORIENTATIONS OF FLEXIBLE BODIES**

Flexible bodies are described with isoparametric hexagonal elements, Section 3.2, hence, only nodal translational degrees of freedom are available to describe relative orientations of flexible bodies with respect to rigid or other flexible bodies. Therefore, imposing joint constraints for relative orientations between two bodies requires additional considerations in order to define a locally fixed coordinate system $\{B; e^B_1, e^B_2, e^B_3\}$ on observers / markers, illustrated in Figure 4. The markers may be attached to arbitrary points in space defined by their relative position $\mathbf{r}_{IB}$ with respect to the inertial system $\{I; e^I_1, e^I_2, e^I_3\}$, or to a body fixed reference system $\{R; e^R_1, e^R_2, e^R_3\}$ with $\mathbf{r}_{IR} = \mathbf{r}_{RB} + \mathbf{r}_{JR}$.

In the case of FE discretized bodies the observer frame can be determined using a coordinate system $\{N; e^N_1, e^N_2, e^N_3\}$, defined by the nodal displacements of three FE nodes $I$, $II$ and $III$ on the body, as illustrated in Figure 5. Nodes $I$, $II$ and $III$ are chosen automatically during initialization being the closest ones to the marker position $\mathbf{r}_{IB}$ and all part of the same element. The coordinate system $\{N; e^N_1, e^N_2, e^N_3\}$ is defined by the following sequence of computations:

Copyright © 2003 by ASME
1. The origin of N is equivalent to the position of node I

\[ \mathbf{r}_{IN} = \mathbf{r}_I = \mathbf{X}_I + \mathbf{U}_I. \]  

(19)

2. Vector \( \mathbf{e}^N_1 \) is defined by the difference vector from node I to node II

\[ \mathbf{e}^N_1 = \frac{\mathbf{r}_{II} - \mathbf{r}_I}{|\mathbf{r}_{II} - \mathbf{r}_I|}. \]  

(20)

3. Vector \( \mathbf{e}^N_2 \) must be orthogonal to \( \mathbf{e}^N_1 \) and in the plane defined by nodes I to III. Vector \( \mathbf{e}^N_2 \) follows as vector orthogonal to both \( \mathbf{e}^N_1 \) and \( \mathbf{e}^N_3 \) i.e. orthogonal to the plane building a right hand system. It is convenient to start with \( \mathbf{e}^N_3 \) that can be determined with the following relation

\[ \mathbf{e}^N_3 = \frac{\mathbf{e}^N_1 	imes (\mathbf{r}_{III} - \mathbf{r}_I)}{|\mathbf{e}^N_1 	imes (\mathbf{r}_{III} - \mathbf{r}_I)|}. \]  

(21)

4. It finally follows

\[ \mathbf{e}^N_2 = \mathbf{e}^N_1 \times \mathbf{e}^N_3. \]  

(22)

Figure 5. COORDINATE SYSTEM DEFINED BY THREE FE–NODES

Further, it is necessary to provide the orientation matrix \( \mathbf{S}_{NI} \), the velocity \( \mathbf{v}_N \) and the angular velocity \( \mathbf{\omega}_N \) of frame \( \{N; \mathbf{e}^N_1, \mathbf{e}^N_2, \mathbf{e}^N_3\} \) with respect to the inertial system. The relative velocity is equal to that of node I, \( \mathbf{v}_N = \mathbf{v}_I \). Also, the orientation matrix does not lead to difficulties, since the base vectors of N are given with respect to the inertial system. It simply follows as

\[ \mathbf{S}_{NI} = \begin{bmatrix} \mathbf{e}^N_1 & \mathbf{e}^N_2 & \mathbf{e}^N_3 \end{bmatrix}. \]  

(23)

For the angular velocity \( \mathbf{\omega}_N \) an approximation is necessary since the nodal information of nodes II and III is used, that have finite distances to node I, while an exact determination of \( \mathbf{\omega}_N \) would require an infinitesimal consideration. The following vector relations are available in order to develop the angular velocity

\[
\begin{align*}
\mathbf{v}_{II} &= \mathbf{v}_I + (\mathbf{r}_I - \mathbf{r}_{II}) \times \mathbf{\omega}_N = \mathbf{v}_I + \mathbf{r}_{II} \times \mathbf{\omega}_N, \\
\mathbf{v}_{III} &= \mathbf{v}_I + (\mathbf{r}_I - \mathbf{r}_{III}) \times \mathbf{\omega}_N = \mathbf{v}_I + \mathbf{r}_{III} \times \mathbf{\omega}_N,
\end{align*}
\]

(24)  
(25)

where the vector product is realized by the skew–symmetric 3 × 3 matrices \( \mathbf{r}_{II} \) and \( \mathbf{r}_{III} \), build from the relative vectors \( \mathbf{r}_I - \mathbf{r}_{II} \) and \( \mathbf{r}_I - \mathbf{r}_{III} \), respectively. Both vector equations form a linear least squares problem with six equations for three unknown quantities

\[ ||\mathbf{A} \cdot \mathbf{\omega}_N - \Delta \mathbf{v}||_2^2 = \min \]  

(26)

with

\[ \mathbf{A} := \begin{bmatrix} \mathbf{r}_{II}^T \\ \mathbf{r}_{III}^T \end{bmatrix} \in \mathbb{R}^{6 \times 3} \quad \text{and} \quad \Delta \mathbf{v} := \begin{bmatrix} \mathbf{v}_I - \mathbf{v}_{II} \\ \mathbf{v}_I - \mathbf{v}_{III} \end{bmatrix} \in \mathbb{R}^6. \]  

(27)

Equation (26) is solved by applying a QR-decomposition with Householder transformations, yielding a mean vector \( \bar{\mathbf{\omega}}_N \).

Since joints are connected to marker frames \( \{B; \mathbf{e}^B_1, \mathbf{e}^B_2, \mathbf{e}^B_3\} \) it is necessary to determine the following quantities for the marker system, which are computed utilizing the information of the nodal fixed system \( \{N; \mathbf{e}^N_1, \mathbf{e}^N_2, \mathbf{e}^N_3\} \), compare Figure 5,

\[ \mathbf{r}_{IB} = \mathbf{r}_{IN} + \mathbf{r}_{NB}, \]  

(28)

\[ \mathbf{v}_B = \mathbf{v}_N + \mathbf{\omega}_N \times \mathbf{r}_{NB}, \]  

(29)

\[ \mathbf{S}_{BI} = \mathbf{S}_{BN} \cdot \mathbf{S}_{NI}, \]  

(30)

\[ \mathbf{\omega}_B = \mathbf{\omega}_N. \]  

(31)

In (30) the orientation matrix \( \mathbf{S}_{BN} \) for transformations from B to N is used in order to describe the constant relative orientation between the user–defined marker frame and the internally defined

Copyright © 2003 by ASME
nodal frame. It can be computed during initialization from the initial orientation of the marker frame $S_{B_i}(t = 0)$ with respect to the inertial system

$$S_{BN} = S_{B_i}(t = 0) \cdot S_{BN}^T(t = 0).$$

(32)

5 DESCRIPTION OF THE GLOBAL SYSTEM – ASSEMBLY OF SUBSYSTEMS

The global system is assembled from $n_s$ subsystems as illustrated in Figure 1. Interactions between different subsystems are described by constraint equations that follow from joint definitions that have to be supplied for all kinds of joints and all combinations: rigid–rigid, rigid–flexible and flexible–flexible.

5.1 Global Equations of Motion

The local equations of motion of the different subsystems, (1), (14) in Section 3, can all be written in the following way with the local generalized coordinates vector $y^i \in \mathbb{R}^{f_i}$

$$M^i (t, y^i) \cdot y^{i'} + k^i (t, y^i, y^i') = g^i (t, y^i, y^i'), \quad i = 1, 2, \ldots, n_s.$$

(33)

The subsystems are connected by $n_j$ joints summarized in the global constraint vector $c \in \mathbb{R}^{n_c}$. The generalized coordinates are assembled in the vector

$$q = [y^1, y^2, \ldots, y^{n_s}] \in \mathbb{R}^q$$

with $n_q = \sum_{i=1}^{n_s} f_i$. (34)

The equations of motion of the global system can, for example, be derived by application of d’Alembert’s principle, which states that the virtual work of reaction forces $\delta W_R$ is zero for the entire system (including the reaction forces between the subsystems), see e. g. [8] or [9],

$$\sum_{i=1}^{n_s} \delta W_R^i = 0,$$

(35)

or equivalently with the virtual work of inertia forces $\delta W_I$ and the virtual work of internal and external applied forces $\delta W_A$

$$\sum_{i=1}^{n_s} \delta W_I^i + \delta W_A^i = 0.$$  

(36)

The expression $\delta W_I^i + \delta W_A^i$ can be described for rigid MBS substructures in Lagrange’s formulation of d’Alembert’s principle [7], resulting in the variations of the local equations of motion (1)

$$\delta W_I^i + \delta W_A^i = \delta y^i \cdot (M^i \cdot y^{i'} + k^i - g^i).$$

(37)

For flexible bodies it follows from Cauchy’s equation of motion for continua [9] and the principle of virtual work after finite element discretization, compare (14) in Section 3.2,

$$\delta W_I^i + \delta W_A^i = \delta U_j \left( M_{jk} \delta U_k^i - g^i_j + \int_{\Omega_0} B_{ijk} F_{ik} \delta u_i dV \right) =: k_j^i.$$  

(38)

where the local generalized coordinates correspond to the nodal displacements of the free body $y^i \equiv U_j \in \mathbb{R}^{f_i}$.

Equation (36) can be split up in $n_s$ rigid MBS subsystems and $n_f$ flexible subsystems, i. e. single flexible bodies. It follows with (37) and (38)

$$\sum_{j=1}^{n_s} \delta y^i_j \cdot (M^i \cdot y^i + k^i - g^i) + \sum_{j=1}^{n_f} \delta U_j \cdot (M^i \cdot \dot{U}^i_j + k^i - g^i) = 0.$$  

(39)

By introduction of the global coordinates vector $q$ from (34) and its variation, equation (39) can be summarized

$$\delta q \cdot (M \cdot \ddot{q} + k - g) = 0 \quad \forall \delta q : C_q^T \cdot \delta q = 0$$

(40)

with the global constraint Jacobian matrix

$$C_q = \frac{\partial c}{\partial q} = \left[ \left( \frac{\partial c^1}{\partial q} \right)^T \left( \frac{\partial c^2}{\partial q} \right)^T \ldots \left( \frac{\partial c^{n_c}}{\partial q} \right)^T \right]^T \in \mathbb{R}^{n_c \times n_q}$$

(41)

the global block diagonal mass matrix $M \in \mathbb{R}^{n_q \times n_q}$

$$M = \text{diag} \left[ M^1 \ldots M^{n_s} | M_{FE}^1 \ldots M_{FE}^{n_f} \right],$$

(42)

and the global vectors $k, g \in \mathbb{R}^q$ that assemble the corresponding local vectors in an analogous structure. Remember that the FE mass matrices $M_{FE}^i$ are constant, as developed in Section 3.2.

In (40) only $f_g = n_q - n_c$ variations of $q$ are independent. By introducing a set of Lagrange multipliers $\lambda \in \mathbb{R}^{n_c}$, (40) can be written in a way valid for arbitrary variations of $q$. This yields the equations of motion of the global multibody system

$$M(t, q) \cdot \ddot{q} + k(t, q, q) - C_q^T(t, q) \cdot \lambda = g(t, q, q)$$

(43)

which form a set of differential algebraic equations (DAE), to be solved together with the constraint vector $c$, or the analytically equivalent first or second time derivatives $\dot{e}$ or $\ddot{e}$. 

Copyright © 2003 by ASME
5.2 Constraint Formulation for the Connection of Substructures

Most of the practically used kinematic constraints can be built by setting algebraic relations between vectors defined on the bodies, as discussed in detail in [1] or [10].

In terms of accuracy and stability of the numerical simulation an index 2 DAE formulation performs very well [11]. This requires the partial derivatives ∂c/∂q and the time derivatives ˙c. In order to determine these derivatives analytically, it is particularly attractive to transform the orientation matrix at the connection points to unit quaternions, e.g., using the algorithm proposed in [12]. Unit quaternions allow for a proper derivation of mathematical relations for observed coordinate frames. Many basic identities and analytical relations are given, e.g., in [10]. They are described as follows

\[ \mathbf{p} = [\mathbf{e}_0, \mathbf{e}] \quad \text{with} \quad \mathbf{e}_0 = \cos \frac{\Phi}{2}, \quad \mathbf{e} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \mathbf{u} \sin \frac{\Phi}{2}, \tag{44} \]

where \( \Phi \) is the rotation angle about a axis described by the unit vector \( \mathbf{u} \) with the additional constraint \( \mathbf{u} \cdot \mathbf{u} = 1 \) or respectively \( \mathbf{p} \cdot \mathbf{p} - 1 = 0 \).

In [13] the required analytical derivatives for constraint vectors using unit quaternions are already derived. For the approach presented in this paper additionally the dependencies of the unit quaternions on the relative coordinates describing the rigid body substructures have to be considered. The necessary relations for rigid substructures are developed in [14].

In the case of flexible multibody dynamics it is further necessary to derive constraint equations that can be applied to both, connections between two flexible bodies and between a rigid and a flexible body. Especially, the extension of existing rigid MBS program systems with large joint libraries to flexible MBS dynamics may lead to an enormous computational effort, since the implementation of the additional constraint equations is necessary for all available joints. A way to overcome this problem has been proposed in [15] by introduction of massless virtual rigid bodies, that can be connected to other rigid bodies utilizing the existing rigid joint library. Therefore, it is only necessary to develop the constraint equations for a single joint that restricts all relative degrees of freedom between the virtual body and a flexible body. A drawback of this approach is, that numerical problems may result due to the fact that zero inertia of the virtual bodies lead to an ill-conditioned mass matrix. However, this problem can be solved by application of a sparse matrix solver. Further, the number of state variables and hence of DAE’s increases, since the virtual bodies have to be treated like additional bodies in the system. Therefore, we decided to avoid the difficulties of the above approach and go for developing and implementing all cases for the limited number of vector conditions.

Constraint Formulation for Joints Connected to Flexible bodies. For the implemented joints in our joint library three basic conditions on vectors serve as building blocks:

1. orthogonality of a pair of body–fixed vectors \( \mathbf{a}_i \) on body \( i \) and \( \mathbf{a}_j \) on body \( j \)

\[ c^{(1)}(\mathbf{a}_i, \mathbf{a}_j) = \mathbf{a}_i \cdot \mathbf{a}_j = 0, \tag{45} \]

2. orthogonality of a body fixed vector \( \mathbf{a}_i \) on body \( i \) and a relative vector \( \mathbf{d}_{ij} \) between bodies \( i \) and \( j \)

\[ c^{(2)}(\mathbf{a}_i, \mathbf{d}_{ij}) = \mathbf{a}_i \cdot \mathbf{d}_{ij} = 0, \tag{46} \]

3. coincidence of points \( B_1 \) on body \( i \) and \( B_2 \) on body \( j \) (spherical constraint)

\[ c^{(3)}(B_1, B_2) = \mathbf{r}_{B_1} - \mathbf{r}_{B_2} = \mathbf{0}. \tag{47} \]

For the system description with an index 2 DAE, the global constraint Jacobian matrix \( \mathbf{C}_q \in \mathbb{R}^{n_r \times n_q} \) and the first time derivative of the global constraint vector \( \dot{\mathbf{c}} \) are required. Since all joints are build from above vector relations, it is only necessary to develop the corresponding entries in \( \mathbf{C}_q \) and \( \dot{\mathbf{c}} \) for all basic relations. The corresponding entries for condition \( c^{(3)} \) are straightforward for flexible bodies, as only nodal displacement data is required.

For further considerations, see the schematic representation of a pair of subsystems as illustrated in Figure 6. In order to describe relative orientations between two bodies with vector conditions it is necessary to have body–fixed marker systems, \( \{ B_1; \mathbf{e}^{B_1}_1, \mathbf{e}^{B_1}_2, \mathbf{e}^{B_1}_3 \} \) and \( \{ B_2; \mathbf{e}^{B_2}_1, \mathbf{e}^{B_2}_2, \mathbf{e}^{B_2}_3 \} \), at the connection points of a joint between two subsystems \( A \) and \( B \). In the following, the case with a rigid MBS subsystem \( A \) and a flexible body as subsystem \( B \) is observed. Both other cases, flexible–rigid and flexible–flexible, lead to similar results.

Figure 6. SCHEMATIC REPRESENTATION OF THE CONNECTION OF TWO SUBSYSTEMS
For discretized flexible bodies that only utilize nodal translational degrees of freedom, a way to define an appropriate marker was determined in Section 4, so that global positions \( r_{B_i} \), rotation matrices \( S_{B_i} \), and translational and angular velocities \( v_{B_i} \) and \( \omega_{B_i} \) are given from equations (28–31).

The rotation matrices at the marker frames \( S_{B_i}(y_A, t) \) and \( S_{B_i}(U_B) \), are transformed to quaternions \( p_A(y_A, t) \) and \( p_B(U_B) \). Note that for flexible bodies the orientation matrices and hence the quaternions do only depend on the nodal displacements.

**ORTHOGONALITY OF A PAIR OF BODY–FIXED VECTORS**

After transformation to quaternions (45) can be written as

\[
e^{c_{n_1}} = e^{c_{n_1}}(p_A(y_A, t), p_B(U_B)).
\]

For its derivative follows

\[
\frac{\partial e^{c_{n_1}}}{\partial q} = \frac{\partial e^{c_{n_1}}}{\partial p_A} \frac{\partial p_A}{\partial q} \frac{\partial e^{c_{n_1}}}{\partial p_B} \frac{\partial p_B}{\partial q}.
\]

The partial derivatives \( \frac{\partial e^{c_{n_1}}}{\partial p_{A,B}} \) are the same as in the rigid case. The only quantity that changes for flexible subsystems is the partial derivative considering the dependencies of the quaternions on the generalized coordinates \( \frac{\partial p_B}{\partial q} \). Since \( p_B(U_B) \) only depends on the local generalized coordinates \( U \) of body \( j \), the structure of \( \frac{\partial p_B}{\partial q} \) has only non–zero entries at the positions corresponding to \( U \) in the global generalized coordinates vector \( q \), e. g.

\[
\frac{\partial p_B}{\partial q} = \begin{bmatrix} 0 & 0 & \cdots & \frac{\partial p_B}{\partial U} & \cdots & 0 & 0 \end{bmatrix}.
\]

In (57) all quantities result from the case rigid–rigid, except the partial derivative \( \frac{\partial r_{B_j}}{\partial q} \). Since \( r_{B_j}(U) \) only depends on the local generalized coordinates \( U \) of body \( j \), the structure of \( \frac{\partial r_{B_j}}{\partial q} \) follows analogous to (50)

\[
\frac{\partial r_{B_j}}{\partial q} = \begin{bmatrix} 0 & 0 & \cdots & \frac{\partial r_{B_j}}{\partial U} & \cdots & 0 & 0 \end{bmatrix}.
\]

In order to derive \( \frac{\partial r_{B_j}}{\partial U} \) consider equation (28)

\[
\frac{\partial}{\partial U} (S_{N1} \cdot r_{NB_j}) = \frac{\partial}{\partial p_N} (S_{N1}(p_N) \cdot r_{NB_j}) \cdot \frac{\partial p_N}{\partial U},
\]

where \( r_{IN} \) is the global position of node \( j \) for the definition of the nodal–fixed frame \( \{ N; e^N_1, e^N_2, e^N_3 \} \), Section 4. The partial derivative \( \frac{\partial r_{IN}}{\partial U} \) leads therefore to a boolean matrix. More complex is the derivative \( \frac{\partial (S_{N1} \cdot r_{NB_j})}{\partial U} \) which can be expressed after transformation of \( S_{N1} \) to unit quaternions \( p_N \) under consideration that \( N_{NB_j} \) is constant

\[
\frac{\partial}{\partial U} (S_{N1} \cdot r_{NB_j}) = \frac{\partial}{\partial p_N} (S_{N1}(p_N) \cdot r_{NB_j}) \cdot \frac{\partial p_N}{\partial U}.
\]

The right term \( \frac{\partial p_N}{\partial U} \) is derived later in the next paragraph, while the left term, a \( 3 \times 4 \) matrix, follows from an identity derived in [13]

\[
\frac{\partial}{\partial p} (S_{N1}(p_N) \cdot r_{NB_j}) = 2^N r_{NB_j} \cdot \dot{p}_N^T + 2 [(e_0 \cdot \dot{e}) \cdot r_{NB_j} - (e_0 \cdot \dot{e}) \cdot r_{NB_j}].
\]
Further, the time derivatives of the condition \( c^p \) are required. With the dependencies in (54) and (55) it follows
\[
\frac{dc^p}{dt} = \frac{d}{dt} \left[ a_i(\mathbf{p}_A(y_A, t)) \cdot \mathbf{d}_{ij}(y_A, \mathbf{U}_R, t) \right] = \left( \frac{\partial a_i}{\partial \mathbf{p}_A} \cdot \mathbf{p}_A \right) \cdot \mathbf{d}_{ij} + a_i(\mathbf{v}_{B_2} - \mathbf{v}_{B_1}), \tag{62}
\]
where the difference to the rigid MBS formulation is given by the vector \( \mathbf{v}_{B_2} \), already determined in (29).

**Derivatives of Quaternions with Respect to the Local Generalized Coordinates of Flexible Bodies:**

In the last paragraphs the partial derivative \( \partial \mathbf{p}/\partial \mathbf{U} \) of quaternions with respect to the generalized coordinates of a flexible body appeared. This quantity is determined in this paragraph.

As stated above, the orientation matrix \( \mathbf{S}_H = \mathbf{S}_H(\mathbf{U}) \) depends for flexible bodies only on the nodal displacement vector \( \mathbf{U} \), and, therefore, after transformation to unit quaternions it follows for the dependencies of the quaternions that \( \mathbf{p} = \mathbf{p}(\mathbf{U}) \). Then, the time derivative of quaternions can be expressed in two ways
\[
\dot{\mathbf{p}} = \frac{\partial \mathbf{p}}{\partial \mathbf{U}} \cdot \dot{\mathbf{U}}, \quad \text{or with (52)} \tag{63}
\]
\[
\dot{\mathbf{p}} = \frac{1}{2} \mathbf{G}^T \cdot \dot{\mathbf{o}} = \mathbf{H} \cdot \dot{\mathbf{U}}. \tag{64}
\]
If it is possible to transform (64) to an expression \( \mathbf{H} \cdot \dot{\mathbf{U}} \), \( \partial \mathbf{p}/\partial \mathbf{U} \) can be found by comparison of coefficients in (63) and (64)
\[
\frac{\partial \mathbf{p}}{\partial \mathbf{U}} = \mathbf{H}. \tag{65}
\]
Since the matrix \( \mathbf{G} = \mathbf{G}(\mathbf{p}) \) given in (53) only depends on \( \mathbf{p} \) and therefore on \( \dot{\mathbf{U}} \), the nodal velocity vector \( \dot{\mathbf{U}} \) must be included in \( \dot{\mathbf{G}} \). This leads to a difficulty, because the angular velocity of the marker frame was approximated by \( \dot{\mathbf{G}} \) in the sense of least squares (26)
\[
\| \mathbf{A} \cdot \dot{\mathbf{o}}_N - \Delta \mathbf{v} \|_2 = \min. \tag{66}
\]
The question now is, how to get the interrelation of \( \dot{\mathbf{G}} \) and \( \dot{\mathbf{U}} \). Equation (66) is solved by application of a QR–decomposition with the orthogonal matrix \( \mathbf{Q} \in \mathbb{R}^{6 \times 6} \) that is build utilizing Householder transformations. It follows
\[
\dot{\mathbf{R}} = \mathbf{Q}^T \cdot \lambda = \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix}, \tag{67}
\]
\[
\Delta \mathbf{v} = \mathbf{Q}^T \cdot \Delta \mathbf{v} = \begin{bmatrix} \Delta \mathbf{v}^* \\ \mathbf{r} \end{bmatrix}, \tag{68}
\]
with the \( 3 \times 3 \) upper triangular matrix \( \mathbf{R} \) and the \( 3 \times 1 \) vector \( \Delta \mathbf{v}^* \). Then (66) yields the following linear set of equations, solved by backward substitution
\[
\mathbf{R} \cdot \dot{\mathbf{o}}_N = \Delta \mathbf{v}^*. \tag{69}
\]

Considering that only \( \Delta \mathbf{v} \) depends on \( \dot{\mathbf{U}} \) and that the dependence on the nodal velocities is linear, it follows
\[
\Delta \mathbf{v} = \begin{bmatrix} \mathbf{v}_I - \mathbf{v}_R \\ \mathbf{v}_J - \mathbf{v}_M \end{bmatrix} = \begin{bmatrix} \mathbf{U}_I - \mathbf{U}_R \\ \mathbf{U}_J - \mathbf{U}_M \end{bmatrix} = \mathbf{V} \cdot \dot{\mathbf{U}}, \tag{70}
\]
where \( \mathbf{U}_I \) are the nodal velocities of nodes \( I \) to \( III \) from the definition of the nodal fixed frame. Since \( \mathbf{U}_I \) are included in the global nodal velocity vector \( \dot{\mathbf{U}} \) of the flexible body, matrix \( \mathbf{V} \) is a boolean matrix. Equation (68) can then be written as
\[
\Delta \mathbf{v} = \mathbf{Q}^T \cdot \mathbf{V} \cdot \dot{\mathbf{U}} = \begin{bmatrix} \mathbf{Q}^T \cdot \mathbf{V}^* \\ \mathbf{Q}^T \cdot \mathbf{V} \end{bmatrix} \cdot \dot{\mathbf{U}}, \tag{71}
\]
which yields with (69)
\[
\dot{\mathbf{o}}_N = \mathbf{R}^{-1} \cdot \mathbf{Q}^T \cdot \mathbf{V}^* \cdot \dot{\mathbf{U}}. \tag{72}
\]
Equations (64) and (72) allow the determination of matrix \( \mathbf{H} \), or rather together with (65), the quantity we were looking for
\[
\frac{\partial \mathbf{p}}{\partial \mathbf{U}} = \mathbf{H} = \frac{1}{2} \mathbf{G}^T \cdot \mathbf{R}^{-1} \cdot \mathbf{Q}^T \cdot \mathbf{V}^*. \tag{73}
\]

**6 EXAMPLE**

Next, an example investigated during verification of the proposed approach is briefly presented, see Figure 7. Details regarding model data and results can be found at http://www.ltm.uni-erlangen.de/Kuebler. The threedimensional motion of a double pendulum consisting of one rigid body and a flexible body of relatively low stiffness is analyzed, where the angular velocity of the rigid body is permanently increased. Even for large centrifugal forces and hence large deformations of the flexible body the constraint conditions for the revolute joint between the two bodies are fulfilled in a sound way.

In order to verify the joint formulation for flexible bodies, the above example was also computed with a relatively high stiffness (steel) for the flexible body and compared to an analogous example with two rigid bodies. Figure 8 illustrates the motion of the lower end of the second body for both models. Shown are the positions \( x_2 \) and velocities \( v_2 \) in horizontal direction, which are in good agreement. Only minor differences to the rigid bodies appear, that are due to the flexibility.

Copyright © 2003 by ASME
Figure 7. FRAMES FROM AN ANIMATION OF THE MECHANISM (LOW STIFFNESS)

Figure 8. COMPARISON OF POSITIONS AND VELOCITIES FOR THE FLEXIBLE MODEL (HIGH STIFFNESS) AND THE RIGID MODEL

7 CONCLUSIONS

The description of flexible bodies as subsystems in a MBS approach offers some favorable characteristics if absolute nodal coordinates are utilized. Absolute coordinates allow for large deformations, translations and rotations and lead for isoparametric elements to constant mass matrices. Further, due to the modular description of the MBS, complex systems can be realized without special considerations under application of constraint equations.

If finite elements with only translational degrees of freedom are used, it is necessary to develop a locally fixed coordinate system in order to describe relative orientations of flexible bodies with respect to other bodies. One way to define such a coordinate system is to use the position of three FE nodes, which performs very well. However, the angular velocity of such a frame can only be determined by approximation which is somehow unsatisfying. To overcome this problem, a different, continuum mechanics based approach may be to utilize the spin tensor which will be realized and compared to the nodal fixed frame in the next step of implementation.

REFERENCES